NONCOMMUTATIVE SYMMETRIC FUNCTIONS AND LAGRANGE INVERSION

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ABSTRACT. We compute the noncommutative Frobenius characteristic of the natural action of the 0-Hecke algebra on parking functions, and obtain as corollaries various forms of the noncommutative Lagrange inversion formula.

1. INTRODUCTION

There are some advantages to interpret the classical Lagrange inversion formula for the reversion of formal power series in terms of symmetric functions (see, e.g., [17], Ex. 24 p. 35, Ex. 25 p. 132, [15] Section 2.4 and [16]). Recall that one possible formulation of the problem is as follows. Given

$$(1) \qquad \varphi(x) = \sum_{n \geq 0} \varphi_n x^n \quad (\varphi_0 \neq 0)$$

find the coefficients c_n of the unique power series

$$(2) \qquad u(t) = \sum_{n \geq 0} c_n t^{n+1}$$

satifying

$$(3) \qquad t = \frac{u}{\varphi(u)}.$$

We can assume without loss of generality that $\varphi_0 = 1$ and that

$$(4) \qquad \varphi(u) = \sum_{n \geq 0} h_n(X) u^n = \prod_{n \geq 1} (1 - u x_n)^{-1} =: \sigma_u(X)$$

is the generating series of the homogeneous symmetric functions of an infinite set of variables X. Indeed, the $h_n(X)$ are algebraically independent, so that $\sigma_u(X)$ is a generic power series.

Now, symmetric functions encode various mathematical objects, and the solution can be interpreted in many ways, for example in terms of characters of the symmetric group. Indeed, in the λ-ring notation, the solution reads

$$(5) \qquad c_n = \frac{1}{n+1} h_n((n+1)X)$$

(recall that $\sigma_t(nX) = \sigma_t(X)^n$, see, e.g., [17] p. 25). On this expression, it is obvious that c_n is Schur positive, in fact, even a positive sum of homogeneous products h_μ, so that it is the Frobenius characteristic of a permutation representation of $\mathfrak{S}_n$. This

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representation is well-known [6]: it is based on the set PF_n of *parking functions* of length n (see below for the definition). The first terms are

$$c_0 = 1, \qquad c_1 = h_1, \qquad c_2 = h_2 + h_{11} \,,$$

(6)
$$c_3 = h_3 + 3h_{21} + h_{111} \,,$$

$$c_4 = h_4 + 4h_{31} + 2h_{22} + 6h_{211} + h_{1111} \,.$$

Now, we have at our disposal noncommutative analogs of the Lagrange inversion formula [4, 19, 1], and a theory of noncommutative symmetric functions [3, 11], known to be related to 0-Hecke algebras in the same way as ordinary symmetric functions are related to symmetric groups [12]. The aim of this note is to clarify the relations between these different topics. We shall first analyze the natural representation of the 0-Hecke algebra on parking functions. This is a projective module, whose q-characteristic noncommutative symmetric function turns out to be the term of degree n in the noncommutative q-Lagrange inversion formula. This allows us to give simple and unified proof of all versions of the noncommutative Lagrange formula [4, 19, 1]. Interpreting the terms as ordered trees leads to closed expressions for the expansion of the solution in various bases. These calculations suggest the introduction of noncommutative analogs of Abel's polynomials, and of an infinite family of combinatorial triangles, which includes classical refinements of the Motzkin, Catalan and Schröder numbers as the first three cases. The action of the 0-Hecke algebra on (k, l)-parking functions is also described.

2. Notations

Our notations for noncommutative symmetric functions will be as in [3, 11]. We recall that the Hopf algebra of noncommutative symmetric functions is denoted by **Sym**, or by **Sym**(A) if we consider the realization in terms of an auxiliary alphabet. Bases of **Sym**$_n$ are labelled by compositions I of n. The noncommutative complete and elementary functions are denoted by S_n and Λ_n, and the notation S^I means $S_{i_1} \cdots S_{i_r}$. The ribbon basis is denoted by R_I. The notation $I \vDash n$ means that I is a composition of n. The conjugate composition is denoted by $I^\sim$.

The graded dual of **Sym** is $QSym$ (quasi-symmetric functions). The dual basis of (S^I) is (M_I) (monomial), and that of (R_I) is (F_I).

The *evaluation* $\mathrm{Ev}(w)$ of a word w over a totally ordered alphabet A is the sequence $(|w|_a)_{a \in A}$ where $|w|_a$ is the number of occurences of a in w. The *packed evaluation* $I = \mathrm{pEv}(w)$ is the composition obtained by removing the zeros in $\mathrm{Ev}(w)$.

The *Hecke algebra* $H_n(q)$ $(q \in \mathbb{C})$ is the $\mathbb{C}$-algebra generated by $n - 1$ elements $T_1, \ldots, T_{n-1}$ satisfying the braid relations and $(T_i - 1)(T_i + q) = 0$. We are interested in the case $q = 0$, whose representation theory can be described in terms of quasi-symmetric functions and noncommutative symmetric functions [12, 2].

The Hopf structures on **Sym** and $QSym$ allows one to mimic, up to a certain extent, the λ-ring notation which is so useful for dealing with ordinary symmetric functions (see [15] for the commutative version and [11] for the noncommutative extension). If A is a totally ordered alphabet, the noncommutative symmetric functions of nA ($n \in \mathbb{Z}$) and $[n]_q A$ (where $[n]_q = \{1 < q < \cdots < q^{n-1}\}$) are defined by

$$(7) \qquad \sigma_t(nA) = \sum_{m \geq 0} t^m S_m(nA) := \sigma_t(A)^n$$

and

$$(8) \qquad \sigma_t([n]_q A) := \sigma_t(A)\sigma_t(qA) \cdots \sigma_t(q^{n-1}A) \,.$$

More generally, noncommutative symmetric functions can be evaluated on any element x of a λ-ring, $S_n(x) = S^n(x)$ being the n-th symmetric power. Recall that x is said *of rank one* (resp. *binomial*) if $\sigma_t(x) = (1 - tx)^{-1}$ (resp. $\sigma_t(x) = (1 - t)^{-x}$). The scalar $x = 1$ is the only element having both properties. We usually consider that our auxiliary variable t is of rank one, so that $\sigma_t(A) = \sigma_1(tA)$.

For each of the noncommutative formulas obtained from representations of the 0-Hecke algebras, we shall give the commutative specializations to the alphabet $A = 1$ ($S_n(1) = 1$ for all n) and to the virtual alphabet $A = \mathbb{E}$, defined by $\sigma_t(\mathbb{E}) = e^t$. This will produce a number of (generally known) combinatorial identities, which can now be traced back to a common source.

3. PERMUTATIONAL 0-HECKE MODULES

3.1. Let $[N] = \{1, \ldots, N\}$ regarded as an ordered alphabet. There is a right action of $H_n(q)$ on $\mathbb{C}[N]^n$ corresponding to the standard right action of $\mathfrak{S}_n$ (see [12]). If $w = a_1 a_2 \cdots a_n$, one sets $w \cdot \sigma_i = a_1 \cdots a_{i+1} a_i \cdots a_n$, and

$$(9) \qquad w \cdot T_i = \begin{cases} w \cdot T_i = & w \cdot \sigma_i & \text{if } a_i < a_{i+1}\,, \\ w \cdot T_i = & q\,w & \text{if } a_i = a_{i+1}\,, \\ w \cdot T_i = & q\,w \cdot \sigma_i + (q-1)\,w & \text{if } a_i > a_{i+1}\,. \end{cases}$$

For $q = 0$, this simplifies as

$$(10) \qquad w \cdot T_i = \begin{cases} w \cdot T_i = & w \cdot \sigma_i & \text{if } a_i < a_{i+1}\,, \\ w \cdot T_i = & 0 & \text{if } a_i = a_{i+1}\,, \\ w \cdot T_i = & -w & \text{if } a_i > a_{i+1}\,. \end{cases}$$

Thus, the image of a word w by an element of $H_n(0)$ is either (up to a sign) a rearrangement of w or 0. In particular, starting from a nondecreasing word v, one obtains all rearrangements of v. These form the basis of a projective $H_n(0)$-module M whose noncommutative characteristic is $\mathbf{ch}(M) = S^I \in \mathbf{Sym} = \mathbf{Sym}(A)$ where I is the packed evaluation of v [12, 2].

The characteristic of the permutation representation $W_n(N) = \mathbb{C}[N]^n$ is easily seen to be

$$(11) \qquad \mathbf{ch}(W_n(N)) = \sum_{I \vDash n} M_I(N) S^I(A) = S_n(NA)$$

by the noncommutative Cauchy identity, since the specialization $M_I(N) := M_I(1^N)$ (1 repeated N times) of the monomial quasi-symmetric function M_I is equal to the number of words of $[N]^n$ with packed evaluation I.

One can do better, and keep track of the sum of the letters, a statistic obviously preserved by the action of $\mathfrak{S}_n$ or $H_n(0)$. We shall normalize it as

$$(12) \qquad \|w\| = \sum_{i=1}^n (a_i - 1) \,.$$

Then,

$$(13) \qquad \sum_{w \in [N]^n, \, \mathrm{pEv}(w)=I} q^{\|w\|} = M_I(1, q, \dots, q^{N-1}) = M_I([N]_q)$$

and we can write down a q-characteristic

$$(14) \qquad \mathbf{ch}_q(W_n(N)) = \sum_{I \vDash n} M_I([N]_q) S^I = S_n([N]_q A) \,.$$

3.2. A *parking function* on $[n] = \{1, 2, \dots, n\}$ is a word $\mathbf{a} = a_1 a_2 \cdots a_n$ of length n on $[n]$ whose non-decreasing rearrangement $\mathbf{a}^\uparrow = a_1' a_2' \cdots a_n'$ satisfies $a_i' \leq i$ for all i. Let PF_n be the set of such words. We are interested in the computation of $G_n(q; A) := \mathbf{ch}_q(\mathrm{PF}_n)$. The first values are:

$$(15) \qquad \begin{aligned}
& G_0 = 1, \qquad G_1 = S_1, \qquad G_2 = S_2 + q S^{11} \,, \\
& G_3 = S^3 + (q + q^2) S^{21} + q^2 S^{12} + q^3 S^{111} \,, \\
& G_4 = S^4 + (q + q^2 + q^3) S^{31} + (q^2 + q^4) S^{22} + q^3 S^{13} + (q^3 + q^4 + q^5) S^{211} \\
& \qquad + (q^4 + q^5) S^{121} + q^5 S^{112} + q^6 S^{1111} \,.
\end{aligned}$$

One can decompose the set of words $w \in [n + r]^n$ according to the length of their maximal parking subword $p(w)$ (which may be empty, and is clearly unique). If $p(w)$ is of length k, the complementary subword can only involve letters greater than $k+1$, and can in fact be any word of $[k + 2, n + r]^{n-k}$. Hence [13],

$$(16) \qquad [n + r]^n = \bigsqcup_{k=0}^n \mathrm{PF}_k \,\shuffle\, [k + 2, n + r]^{n-k} \,.$$

Taking the q-characteristic of the underlying permutational 0-Hecke modules, and remembering that shuffling over disjoint alphabets amounts to inducing representations, we obtain

$$(17) \qquad S_n([n + r]_q A) = \sum_{k=0}^n \mathbf{ch}_q(\mathrm{PF}_k) q^{(k+1)(n-k)} S_{n-k}([n + r - k - 1]_q A)$$

which allows us to extract the generating series of $G_n(q; A) := \mathbf{ch}_q(\mathrm{PF}_n)$. Indeed, writing

$$(18) \qquad (k + 1)(n - k) = \binom{n+1}{2} - \binom{k+1}{2} - \binom{n-k}{2}$$

and

$$(19) \qquad F^{(r)}(x, q; A) = \sum_{n \geq 0} x^n q^{-\binom{n}{2}} S_n([n+r]_q A)$$

we arrive at

Theorem 3.1. *The generating series of* $G_n(q; A) := \mathbf{ch}_q(\mathrm{PF}_n)$ *is*

$$(20) \qquad G(x, q; A) := \sum_{n \geq 0} x^n q^{-\binom{n+1}{2}} G_n(q; A) = F^{(r)}(xq^{-1}, q; A) \ F^{(r-1)}(x, q; A)^{-1}.$$

In particular, this expression is independent of r, a fact which is not easily derived by mere algebraic manipulations.

We can let r tend to infinity, and obtain the simpler form

$$(21) \qquad G(x, q; A) = F(xq^{-1}, q; A) F(x, q; A)^{-1}$$

where

$$(22) \qquad F(x, q; A) = \sum_{n \geq 0} x^n q^{-\binom{n}{2}} S_n \left(\frac{A}{1-q} \right).$$

Example 3.2. Let us take the specialization $A = \mathbb{E}$, where the "exponential alphabet" $\mathbb{E}$ is defined by $\sigma_t(\mathbb{E}) = e^t$ (that is, $S_n(\mathbb{E}) = 1/n!$). Then

$$(23) \qquad \sigma_t \left(\frac{\mathbb{E}}{1-q} \right) = \exp \left(\frac{t}{1-q} \right)$$

and we recover Gessel's formula for the sum enumerator of parking functions ([4], see also [22], Ex. 5.48.b and 5.49.c pp. 94-95).

Example 3.3. If we take $A = 1$, so that $\sigma_t(1) = (1 - t)^{-1}$, we have

$$(24) \qquad S_n \left(\frac{1}{1-q} \right) = \frac{1}{(q)_n}$$

and replacing q by $1/q$ and x by -1 we recognize in $F(-1, 1/q; 1)$ and $F(-q, 1/q; 1)$ the left-hand sides of the Rogers-Ramanujan identities. We have in fact an infinity of different expressions of the Ramanujan function $F(-qx, 1/q; 1)F(-x, 1/q; 1)^{-1}$ as $F^{(r)}(-qx, 1/q; 1)F^{(r-1)}(-x, 1/q; 1)^{-1}$. The case $r = 1$ is obtained in [19] (precisely as an application of noncommutative Lagrange inversion).

4. The functional equation

We shall now see that $G(x, q; A)$ solves a functional equation, and recover the noncommutative q-Lagrange formula in this way. For later convenience, let us first change q into $1/q$ and consider

$$(25) \qquad H(x, q; A) := G(x, q^{-1}; A) = E(qx)E(x)^{-1}$$

where

$$(26) \qquad E(x) = E(x, q; A) = \sum_{n \geq 0} x^n q^{\binom{n}{2}} S_n \left(\frac{A}{1-q^{-1}} \right).$$

Then, $H(q^{k-1}x) = E(q^k x)E(q^{k-1}x)^{-1}$, so that

(27) $$H^{(n)}(x) := H(q^{n-1}x)H(q^{n-2}x)\cdots H(x) = E(q^n x)E(x)^{-1},$$

and we can write

$$\sum_{n\geq 0} x^n q^{\binom{n+1}{2}} S_n(A) H^{(n)}(x) = \sum_{n\geq 0} x^n q^{\binom{n+1}{2}} S_n(A) E(q^n x) E(x)^{-1}$$

$$= \sum_{n\geq 0} x^n q^{\binom{n+1}{2}} S_n(A) \sum_{m\geq 0} q^{\binom{m}{2}} S_m \left(\frac{A}{1-q^{-1}}\right) E(x)^{-1}$$

$$= \sum_{N\geq 0} x^N q^{\binom{N}{2}} S_N \left(qA + \frac{A}{1-q^{-1}}\right) E(x)^{-1}$$

$$= \sum_{N\geq 0} x^N q^{\binom{N+1}{2}} S_N \left(\frac{A}{1-q^{-1}}\right) E(x)^{-1}$$

$$= E(qx)E(x)^{-1} = H(x).$$

The powers of q can be absorbed in the products if we set $K(x) = xqH(x)$. Finally, we obtain

Theorem 4.1. *The series $K(x) = K(x, q; A) = xqG(x, q^{-1}; A)$ solves the functional equation of the noncommutative q-Lagrange formula of* [4, 19]

(28) $$K(x) = qx \sum_{n\geq 0} S_n(A) \cdot K^{(n)}(x).$$

One has
(29)
$$K(x) = xq + x^2 q^2 S_1 + x^3(q^4 S_2 + q^3 S^{11}) + x^4(q^7 S_3 + (q^5 + q^6)S^{21} + q^5 S^{12} + q^4 S^{111})$$
$$+ x^5(q^{11} S_4 + (q^8 + q^9 + q^{10})S^{31} + (q^7 + q^9)S^{22} + q^8 S^{13} + (q^6 + q^7 + q^8)S^{211}$$
$$+ (q^6 + q^7)S^{121} + q^6 S^{112} + q^5 S^{1111}) + \cdots$$

In particular, $g(A) = G(1,1;A) = \sum \mathbf{ch}(\mathrm{PF}_n)$ is the unique solution of

(30) $$g = \sum_{n\geq 0} S_n g^n,$$

with $S_0 = 1$. The first terms are

$$g_0 = 1, \qquad g_1 = S_1, \qquad g_2 = S_2 + S^{11},$$

(31) $$g_3 = S_3 + 2S^{21} + S^{12} + S^{111},$$

$$g_4 = S_4 + 3S^{31} + 2S^{22} + 3S^{13} + 3S^{211} + 2S^{121} + S^{112} + S^{1111}.$$

Note that g_i is obtained by setting $q = 1$ in (15), that (30) is (29) for $q = x = 1$ and that one recovers (6) by assuming that the S_i commute.

The solution of [4, 19] is obtained by taking $r = 1$ in Formula (20). The commutative image gives various forms of the Garsia-Gessel q-Lagrange formula.

5. THE GENERAL NONCOMMUTATIVE LAGRANGE INVERSION FORMULA

5.1. **Nondecreasing parking functions.** The versions of [4] and [19] on the non-commutative inversion formula deal with the slightly more general functional equation

$$(32) \qquad f = S_0 + S_1 f + S_2 f^2 + S_3 f^3 + \cdots ,$$

where S_0 is another indeterminate which does not necessarily commute with the other ones. The solution can be expressed in the form

$$(33) \qquad f_n = \sum_{\pi \in \mathrm{NDPF}_n} S^{\mathrm{Ev}(\pi) \cdot 0} ,$$

where NDPF_n denotes the set of nondecreasing parking functions on $[n]$. For example,

$$(34) \qquad f_0 = S_0 , \ f_1 = S_1 S_0 = S^{10} , \ f_2 = S^{110} + S^{200}$$

$$(35) \qquad f_3 = S^{1110} + S^{1200} + S^{2010} + S^{2100} + S^{3000} ,$$

the nondecreasing parking functions giving f_3 being (in this order) 123, 122, 113, 112, 111.

5.2. **Dyck words.** Here is an amusing way to prove Formula (33), inspired by one of the examples of [19]. If we denote by D the sum of all Dyck words (1 being the empty word)

$$(36) \quad D(a,b) = 1 + ab + aabb + abab + aaabbb + aabbab + aababb + abaabb + ababab + \cdots$$

which can be defined by the functional equation

$$(37) \qquad D = 1 + aDbD ,$$

then, the series $f = Db$ satifies (32), with

$$(38) \qquad S_n = a^n b .$$

Indeed, iterating (36), we have

$$(39) \qquad \begin{aligned} Db &= b + a(Db)(Db) = b + a[b + a(Db)(Db)](Db) \\ &= b + ab(Db) + aab(Db)(Db) + aaab(Db)(Db)(Db) + \cdots , \end{aligned}$$

so that we know the solution of (32) in this particular case. But the particular case is generic: $S = \{a^n b \mid n \geq 0\}$ is a prefix code, and the S_n defined by (38) are algebraically independent. The general solution (33) is then obtained by decomposing the words of Db on the code S.

This expression being granted, the other version of the solution (as a quotient of series) is obtained directly from (16) as above.

In [19], the specialization $S_n = \frac{1}{n!} a^n b$ is also considered, leading to what the authors have called noncommutative inversion polynomials.

5.3. **Trees.** Alternatively, Formula (32) can be interpreted as a sum over ordered trees. Let us set $c = S_0$, $d_n = S_n$, and interpret d_n as the symbol of an n-ary operation in Polish notation, so that for example

(40) $$f_3 = d_1 d_1 d_1 c + d_1 d_2 cc + d_2 cd_1 c + d_2 d_1 cc + d_3 ccc$$

is the Polish notation for

(41) $$d_1(d_1(d_1(c))) + d_1(d_2(c,c)) + d_2(c,d_1(c)) + d_2(d_1(c),c) + d_3(c,c,c)$$
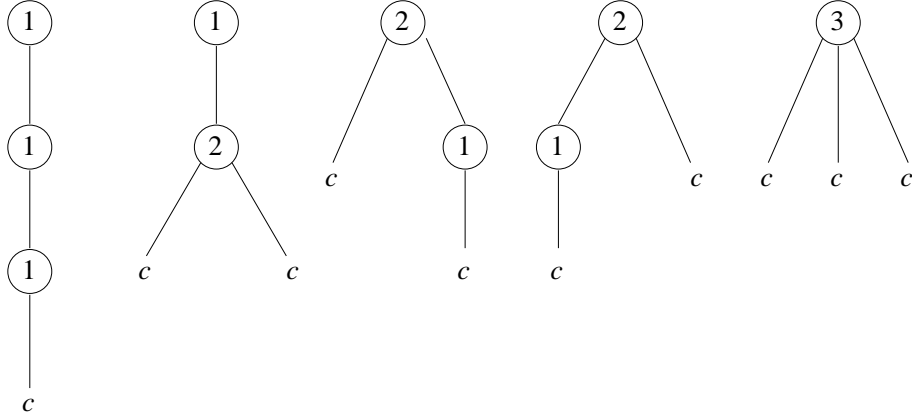
and corresponds to the five ordered trees of Figure 1.

FIGURE 1. The five ordered trees corresponding to f_3.

This implies an expression of the coefficients δ_I defined by

(42) $$g_n = \mathrm{ch}(\mathrm{PF}_n) = \sum_{I \models n} \delta_I S^I$$

since g_n is obtained from f_n by setting $c = 1$ in (30). Indeed, given a tree T, define its *skeleton* as the tree obtained by removing the leaves c and labeling the internal vertices with their arity. Given the skeleton S of a tree T, define its 0-*composition* $I_0(S)$ as the sequence formed by the values of the labels of the vertices of S read in prefix order.

For example, one finds on Figure 2 a tree and its skeleton. The corresponding 0-composition is $(3, 2, 4, 2)$.

The number of trees with skeleton S is obviously

(43) $$\prod_{k=1}^{p} \binom{i_k}{a_k}$$

where $I_0(S) = (i_1, \ldots, i_p)$ and a_k is the arity of the k-th vertex of the tree S, numbered in prefix order.

For example, there are 16 trees whose skeleton have $(3, 1, 2, 1)$ as 0-composition as one can check on Figure 3.

Let $I = (i_1, \ldots, i_p)$ be a composition of n. We are now in a position to compute δ_I. Indeed, the coefficient of S^I in g_n is equal to the number of ordered trees on $n+1$
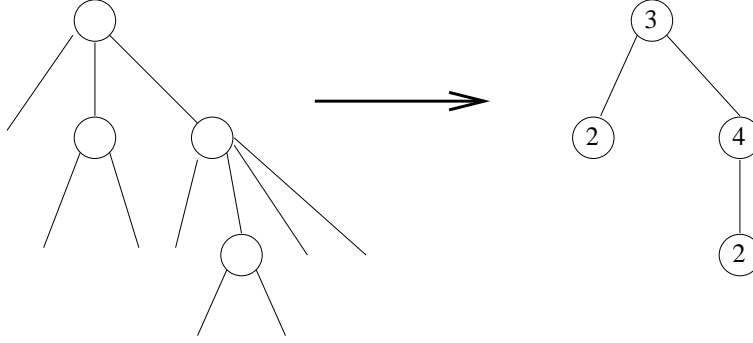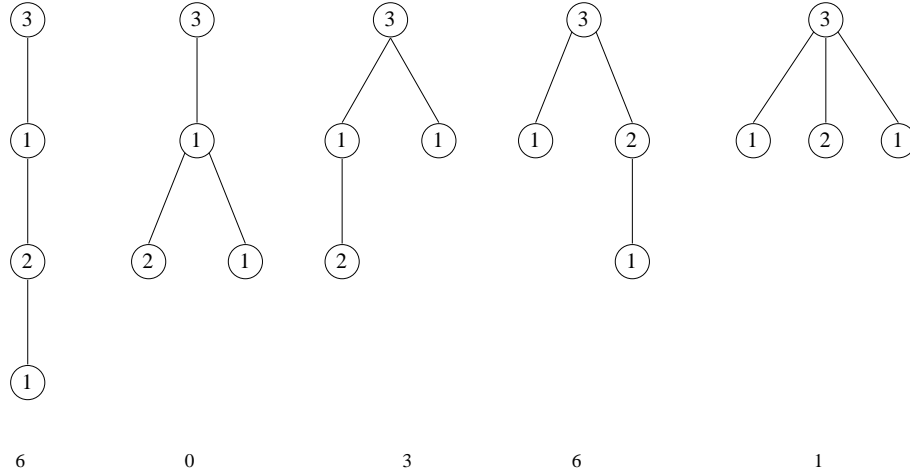
FIGURE 2. A tree and its skeleton.

FIGURE 3. Skeletons having $(3, 1, 2, 1)$ as 0-composition, and the number of trees with those skeletons.

vertices whose sequence of non-zero arities in prefix order is I. The skeletons of these trees are the ordered trees on p vertices labeled by the elements of I in prefix order. The sequences of arities of the skeletons in prefix order are all sequences $(a_1, \ldots, a_p)$ such that $a_1 + \cdots + a_j \geq j$ for $j < p$ and equal to $p - 1$ if $j = p$. Thus $a_p = 0$ and $a_1 + \cdots + a_{p-1} = p - 1$ so that

$$(44) \qquad \delta_I = \sum_{(a_1, \ldots, a_{p-1})} \prod_{k=1}^{p-1} \binom{i_k}{a_k},$$

where the sum is taken over the set of sequences $(a_1, \ldots, a_{p-1})$ such that $a_1 + \cdots + a_j \geq j$ for all j and $a_1 + \cdots + a_{p-1} = p - 1$.

6. Noncommutative formal diffeomorphisms

6.1.   Another form of the noncommutative Lagrange inversion has been obtained by Brouder-Frabetti-Krattenthaler [1]. It is stated in the form of an explicit formula for the antipode of the Hopf algebra $\mathcal{H}^{\mathrm{dif}}$ of "formal diffeomorphisms". As an associative

algebra, $\mathcal{H}^{\mathrm{dif}}$ can be identified with **Sym** by means of the correspondence $a_n = S_n = S_n(A)$. The coproduct can then be expressed as

$$(45) \qquad \Delta^{\mathrm{dif}} S_n(A) = \sum_{k=0}^{n} S_k(A) \otimes S_{n-k}((k+1)A).$$

In this notation, computing the antipode amounts to find a series

$$(46) \qquad h(A) = \sum_{n \geq 0} b_n$$

where $b_n \in \mathbf{Sym}_n(A)$, such that

$$(47) \qquad 1 = \sum_{n \geq 0} S_n(A) h(A)^{n+1}.$$

Hence, $h(A)$ satisfies the functional equation

$$(48) \qquad h(A)^{-1} = \sum_{n \geq 0} S_n(A) h(A)^n,$$

differing from that of Gessel and Pak-Postnikov-Retakh, which reads

$$(49) \qquad g(A) = \sum_{n \geq 0} S_n(A) g(A)^n.$$

However, the difference is not that big, since we have

Theorem 6.1. *The relation between the noncommutative symmetric series $h(A)$ and $g(A)$ respectively defined by (48) and (49) is*

$$(50) \qquad h(A) = g(-A).$$

Proof – This is a good illustration of the power of the "noncommutative λ-ring notation". Using the expression of $g(A)$ given by putting $q = 1$ in (20), we can write

$$(51) \qquad g(-A)^n = F(-n)F(0)^{-1} \quad (n \in \mathbb{Z})$$

where

$$(52) \qquad F(x) = \sum_{m \geq 0} S_m((x-m)A).$$

Hence,

$$\begin{aligned}
\sum_{n \geq 0} S_n(A) g(-A)^n &= \sum_{n \geq 0} S_n(A) F(-n) F(0)^{-1} \\
&= \sum_{n \geq 0} S_n(A) \sum_{m \geq 0} S_m((-m-n)A) F(0)^{-1} \\
&= \sum_{N \geq 0} \sum_{m+n=N} S_n(A) S_m((-m-n)A) F(0)^{-1} \\
&= \sum_{N \geq 0} S_N((-N+1)A) = F(1) F(0)^{-1} = g(-A)^{-1}.
\end{aligned}$$

Remark 6.2. This calculation works as well for the q-analog, and allows one to compute the antipode of the q-deformed coproduct

$$(53) \qquad \Delta_q^{\mathrm{dif}} S_n(A) = \sum_{k=0}^{n} S_n(A) \otimes S_{n-k}([k]_q A) .$$

For $q = 0$, this is the usual coproduct of **Sym**. We have therefore an interpolation between the two structures, the combinatorics being governed by the q-Lagrange formula, hence by parking functions.

6.2. **Trees.** One can give for $h(A)$ a combinatorial interpretation analogous to (33). Starting from the generalized inversion problem

$$(54) \qquad f^{-1} = S_0 + S_1 f + S_2 f^2 + \dots,$$

we recast it in the form

$$(55) \qquad f = c + d_1 f^2 + d_2 f^3 + \dots,$$

setting $c = S_0^{-1}$ and $d_n = -S_0^{-1} S_n$. Solving recursively for f_0, f_1, ..., we find

$$(56) \qquad f_0 = c, \quad f_1 = d_1 cc, \quad f_2 = d_1 c d_1 cc + d_1 d_1 ccc + d_2 ccc ,$$

and we can now interpret each d_i as the symbol of an $(i+1)$-ary operation in Polish notation. Then, f_n is the sum of Polish codes of ordered trees with no vertex of arity 1 on $2n+1$ vertices (or Schröder bracketings of the words c^{n+1}) as one can check on Figure 4. From this, we can easily recover Formula (2.21) of [1]. This amounts to set

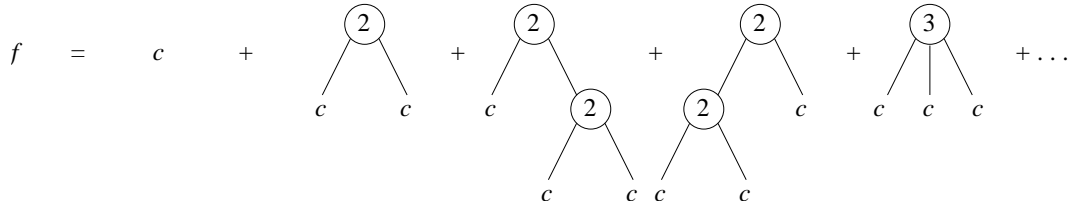FIGURE 4. The terms f_0, f_1, f_2 expressed as a sum of ordered trees.

$c = 1$, that is, solving

$$(57) \qquad h = 1 + d_1 h^2 + d_2 h^3 + \dots,$$

as

$$(58) \qquad h_n = \sum_{I \models n} \lambda_I d^I .$$

We proceed as in the previous section. Given the skeleton S of a tree T, define its 1-*composition* $I_1(S)$ as the sequence of values of the labels of the vertices of S minus

1 in prefix order. Then, thanks to Equation (43), the number of trees with skeleton S is

$$
(59) \qquad \prod_{k=1}^{p} \binom{i_k + 1}{a_k}
$$

where $I_1(S) = (i_1, \ldots, i_p)$ and a_k is the arity of the k-th vertex of S, numbered in prefix order. For example, there are 34 trees whose skeleton have $(1, 3, 1, 1)$ as associated 1-composition as one can check on Figure 5.
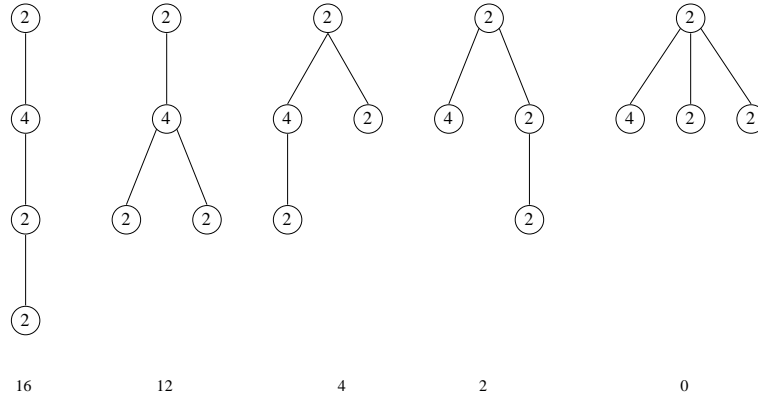

FIGURE 5.

Let $I = (i_1, \ldots, i_p)$ be a composition of n. The coefficient of S^I in h_n is equal to the number of ordered trees on $2n + 1$ vertices whose sequence of non-zero arities minus one in the prefix reading is I. As before, the skeletons of these trees are the ordered trees on p vertices labeled by one plus the elements of I in prefix order. The sequences of arities of the skeletons are the same as before so that

$$
(60) \qquad \lambda_I = \sum_{(a_1, \ldots, a_{p-1})} \prod_{k=1}^{p-1} \binom{i_k + 1}{a_k}
$$

where the sum is taken over the set of a_k such that $a_1 + \cdots + a_j \geq j$ for all j and $a_1 + \cdots + a_{p-1} = p - 1$. This is Formula (2.23) of [1]. In this presentation, it is clear that the sum is over a Catalan set, here the ordered trees.

7. Explicit expressions in various bases

7.1.   We shall now compute the coefficients of the expansions of g (or h as well) in the bases (R_I) and (Λ^I) of **Sym**.

The expansion on ribbons can be given for the q-analogs. Let $\mathrm{QRP}(I)$ be the set of parking quasi-ribbons of shape I (see [18]), and let

$$
(61) \qquad c_I(q) = \sum_{\mathbf{a} \in \mathrm{QRP}(I)} q^{\|\mathbf{a}\|}.
$$

Then, since two words with the same evaluation are hypoplactically equivalent iff the inverses of their standardized have the same descents,

$$(62) \qquad \mathrm{ch}_q(\mathrm{PF}_n) = G_n(q; A) = \sum_{I \vDash n} c_I(q) R_I(A) \,.$$

For example,

$$
\begin{aligned}
G_3(q; A) &= S^3 + (q + q^2)S^{21} + q^2 S^{12} + q^3 S^{111} \\
(63) \qquad &= (1 + q + 2q^2 + q^3)R_3 + (q + q^2 + q^3)R_{21} + (q^2 + q^3)R_{12} + q^3 R_{111}
\end{aligned}
$$

For $q = 1$, this expansion presents a remarkable symmetry. The expansion on elementary functions is given by the same formula as the expansion on ribbons, up to sign and conjugation of the compositions:

$$(64) \qquad g_n(A) = \sum_{I \vDash n} (-1)^{n-l(I)} c_{I^\sim} \Lambda^I \,.$$

For example,

$$
\begin{aligned}
(65) \quad g_3(A) &= \Lambda^3 - 3\Lambda^{21} - 2\Lambda^{12} + 5\Lambda^{111} \,, \\
(66) \quad g_4(A) &= -\Lambda^4 + 4\Lambda^{31} + 3\Lambda^{22} + 2\Lambda^{13} - 9\Lambda^{211} - 7\Lambda^{121} - 5\Lambda^{112} + 14\Lambda^{1111} \,.
\end{aligned}
$$

This symmetry is equivalent to the invariance of g under the linear involution of **Sym** defined by

$$(67) \qquad \nu : \; S^I \longmapsto S^{I^\sim} \,,$$

as one can check on Equation (31). Indeed,

Lemma 7.1. *On the ribbon basis, ν is given by*

$$(68) \qquad \nu(R_I) = (-1)^{l(I)-1} \Lambda^{I^\sim} \,.$$

Proof – The image of the Cauchy kernel $\sigma_1(XA)$ by ν is

$$\sum_I F_I \nu(R_I) = \sum_I M_I \nu(S^I) = \sum_I M_I S^{I^\sim} = \sum_I M_{I^\sim} S^I$$

$$= \sum_I \sum_{J \leq I} M_{I^\sim} R_J = \sum_I M_{I^\sim} \sum_{J^\sim \geq I^\sim} R_{J^\sim}$$

$$= \sum_I M_{I^\sim} \sum_{J^\sim \geq I^\sim} \sum_{K \leq J} (-1)^{l(J)-l(K)} \Lambda^K$$

(69) $$= \sum_K \left(\sum_{J \geq K} (-1)^{l(J)} \sum_{I^\sim \leq J^\sim} M_{I^\sim} \right) \Lambda^K$$

$$= \sum_K \left(\sum_I M_{I^\sim} \sum_{K \leq J \leq I} (-1)^{l(J)} \right) (-1)^{l(K)} \Lambda^K$$

$$= \sum_K \left(\sum_{I \geq K} M_{I^\sim} (-1)^{l(K)} \right) \Lambda^K = \sum_K F_{K^\sim} (-1)^{l(K)} \Lambda^K$$

$$= \sum_K F_K (-1)^{l(K)-1} \Lambda^{K^\sim} .$$

■

7.2. **An involution.** Actually, the ν-invariance of g follows from a stronger property. As we have seen, the solution f of the general inversion problem (32)

(70) $\qquad f = S^0 + S^{10} + S^{200} + S^{110} + S^{1110} + S^{1200} + S^{2010} + S^{2100} + S^{3000} + \cdots$

can be interpreted as the formal sum of all nondecreasing parking functions. We will now prove that there exists a canonical involution ι on these which satisfies

(71) $$\mathrm{pEv}(\iota(\pi)) = \mathrm{pEv}(\pi)^\sim .$$

To simplify the presentation, we shall identify a nondecreasing parking function with its evaluation. More precisely, define a *generalized composition* as a composition where zeros are allowed. The composition obtained by removing all zeros is called the *corresponding composition*. A generalized composition I of n is of *parking type* iff it is of length $n+1$ and $i_1 + \ldots + i_k \geq k$ for all k in $[1,n]$. In other words, the set of generalized compositions of parking type is the set of evaluations of parking functions with an appropriate number of trailing zeros.

Before describing the involution on generalized compositions of parking type, we need some more structure on the set of elements having the same packed evaluation. For each composition I of n, build a directed graph Γ_I with vertex set given by generalized compositions of parking type with corresponding composition I and an arrow $J \longrightarrow J'$ iff J' is obtained from J by exchanging two consecutive parts of J, j_i and j_{i+1} so that j_i or j_{i+1} is 0, an operation reminescent of Hivert's quasi-symmetrizing action [7]. For example, Γ_{331} and Γ_{21211} are given on Figure 6. Γ_I can be seen as an initial interval of a permutohedron: consider the word $K = (0^{n+1-l(I)}$
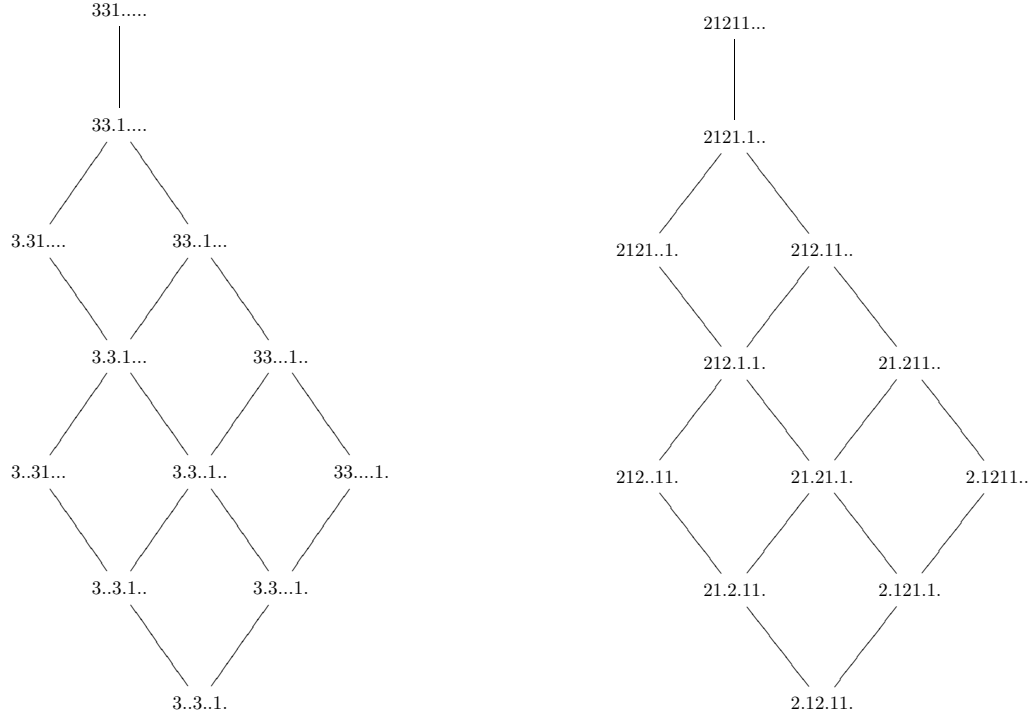
FIGURE 6. The graphs Γ_{331} and Γ_{21211} (0s are represented by dots).

and the shuffle $S = I \shuffle K$. To these elements corresponds naturally an element of the shuffle $S' = (123 \cdots l(I)) \shuffle (l(I)+1 \cdots n+1)$. Then if one restricts to the elements of S that are the evaluation of a nondecreasing parking function, for any such element s, all the permutations smaller than the corresponding element s' in S' correspond to evaluations of nondecreasing parking functions: indeed, this means that if J is of parking type, all generalized compositions obtained from J by moving zeros to the right also are of parking type, which is obvious. Now there is only one minimal element, the concatenation of I and K, and only one maximal element, the evaluation where any non-zero entry i, except for the last one, is followed by exactly $i-1$ zeros before the next non-zero entry: no successor of this element is the evaluation of a parking function and all other elements have at least one successor of this type.

We are now in a position to describe the involution on generalized compositions of parking type which induces the conjugation on the underlying compositions.

Algorithm 7.2. *Let J be a generalized composition of parking type.*
- *Let J' be the tuple obtained by reading J from right to left.*
- *Compute the conjugate C of the corresponding composition of J,*
- *fill the zero slots of J' by the parts of C,*
- *replace by 0 the nonzero parts of J'.*
- *This is the output, $\iota(J)$.*

For example, if $J = (2,1,1,0,1,2,0,2,0,0)$, then $J' = (0,0,2,0,2,1,0,1,1,2)$, the corresponding composition being $(2,1,1,1,2,2)$ and its conjugate $(1,2,5,1)$. We

insert the four parts of this last composition into the zero slots of J', putting zeros at the other places and get $(1, 2, 0, 5, 0, 0, 1, 0, 0, 0)$.

Lemma 7.3. *The previous algorithm is an involution on generalized compositions of parking type, sending maximal elements of graphs to maximal elements.*

Proof – The algorithm is an involution since the conjugation of compositions is one, so we only have to prove that the output is of parking type if the input is. By construction of Γ_I, it is sufficient to prove that the image of the bottom element of Γ_I is of parking type. Thanks to its characterization, it is obvious that this bottom element is sent to the bottom element of $\Gamma_{I^\sim}$ by our involution.   ∎

Theorem 7.4. *The graphs Γ_I and $\Gamma_{I^\sim}$ associated to mutually conjugate compositions of n are isomorphic. Moreover, if one labels the edges by i when one exchanges the letters in positions i and $i+1$, then the labels of the edges are exchanged by the involution $i \leftrightarrow n+1-i$.*

Proof – The graph Γ_I corresponds to a part of the shuffle $I \shuffle 0^{n+1-l(I)}$ whereas the graph $\Gamma_{I^\sim}$ corresponds to a part of the shuffle $I^\sim \shuffle 0^{n+1-l(I^\sim)}$. It is known that $l(I) + l(I^\sim) = n+1$ so that both graphs correspond to parts of a shuffle of an element of length $l(I)$ with an element of length $l(I^\sim)$. Moreover, given the definition of the edges of both graphs, an edge labelled i between P and P' proves that there is an edge labelled $n+1-i$ between $\iota(P)$ and $\iota(P')$. So both graphs are isomorphic.   ∎

For example, the two graphs on Figure 6 corresponding to 331 and 331$^\sim$ are indeed isomorphic.

7.3. **A representation theoretical interpretation.** In fact, nondecreasing parking functions form a sub-semigroup of the semigroup of all endofunctions of $[n]$. Its representation theory has been investigated by Hivert and Thiéry [9], and it follows from their work that the graphs Γ_I (now seen on nondecreasing parking functions instead of generalized compositions of parking type) encode the indecomposable projective modules P_I of the semigroup algebra $\mathcal{C}_n = \mathbb{C}[\mathrm{NDPF}_n]$. Indeed, these modules are parametrized by compositions of n, and each P_I has a basis $(b_\pi)_{\pi \in \mathrm{NDPF}_I}$, such that if one denotes by e_i the generator mapping $i+1$ to i and leaving invariant all other j, $e_i \circ b_\pi = b_{\pi'}$ iff $\pi \xrightarrow{i} \pi'$ and $e_i \circ b_\pi = 0$ otherwise. Thus, on the one hand, the coefficients δ_I of the expansion

$$(72) \qquad\qquad g_n = \mathbf{ch}(\mathrm{PF}_n) = \sum_{I \vDash n} \delta_I S^I$$

are the dimensions $\delta_I = \dim P_I$ of the indecomposable projective modules of $\mathcal{C}_n$. On the other hand, the noncommutative symmetric functions S^I are the characteristics of the permutational modules of $H_n(0)$, which are projective, but decomposable for $I \neq (n)$. As also shown in [9], these permutational modules are in fact the indecomposable projective modules for a larger algebra, the *Hecke-symmetric* algebra $\mathrm{H}\mathfrak{S}_n$. One can check that the right action of $\mathrm{H}\mathfrak{S}_n$ on $\mathbb{C}\,\mathrm{PF}_n$ and the left action of $\mathcal{C}_n$ (by composition

$\pi \circ \mathbf{a}$) commute with each other, so that the expression (72) of $\mathbf{ch}(\mathrm{PF}_n)$ reflects the decomposition of $\mathbb{C}\,\mathrm{PF}_n$ as a $(\mathcal{C}_n, \mathrm{H}\mathfrak{S}_n)$-bimodule.

The coefficients (λ_I) of the ribbon expansion can be similarly interpreted as the dimensions of the projective modules of the commutant of $H_n(0)$ in $\mathbb{C}\,\mathrm{PF}_n$, an algebra $\mathcal{D}_n$ having as dimension the Schröder number s_n and containing $\mathcal{C}_n$.

8. NONCOMMUTATIVE ABEL IDENTITIES

8.1. Abel's generalization of the binomial identity can be stated as

$$(73) \qquad p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y),$$

that is, the Abel polynomials

$$(74) \qquad p_n(x) := x(x+n)^{n-1}$$

form a sequence of binomial type.

Let $\mathbb{E}$ be the exponential alphabet. If we set $g_n = \mathbf{ch}(\mathrm{PF}_n)$, we have $g_n(t\mathbb{E}) = (n+1)^{n-1}t^n/n!$, and it follows from (16) that

$$(75) \qquad g^r = \sum_{n \geq 0} \mathbf{ch}(\mathrm{PF}_n^{(r)}) =: g^{(r)}$$

where

$$(76) \qquad g_n^{(r)}(t\mathbb{E}) = \frac{t^n}{n!}|\mathrm{PF}_n^{(r)}| = \frac{t^n}{n!} r(r+n)^{n-1}$$

and $\mathrm{PF}_n^{(r)}$ is the set of words $\mathbf{a}$ whose nondecreasing rearrangement satisfies $\mathbf{a}_i^{\uparrow} \leq i + r - 1$. This is because of the self-evident generalization of (16)

$$(77) \qquad \mathrm{PF}_n^{(r)} = \bigsqcup_{k=0}^{n} \mathrm{PF}_k^{(j)} \,\text{⧢}\, \mathrm{PF}_{n-k}^{(r-j)},$$

for all j in $[1, r-1]$, which implies in particular

$$(78) \qquad \mathbf{ch}(\mathrm{PF}_n^{(r)}) = \sum_{k=0}^{n} \mathbf{ch}(\mathrm{PF}_k)\mathbf{ch}(\mathrm{PF}_{n-k}^{(r-1)}).$$

Hence, the $t\mathbb{E}$-specialization of g^x is the exponential generating function of Abel's polynomials, and Abel's identity amounts to the obvious equality $g^x g^y = g^{x+y}$.

We can therefore define the noncommutative Abel polynomial $P_n(x; A)$ as the term of degree n in $g(A)^x$. It can be computed directly using the binomial expansion of

$g(A)^x = (1+U)^x = \sum_n \binom{x}{n} U^n$. For example,

$$P_1(x; A) = xS^1,$$

$$P_2(x; A) = xS^2 + \frac{x(x+1)}{2}S^{11},$$

$$P_3(x; A) = xS^3 + \frac{x(x+3)}{2}S^{21} + \frac{x(x+1)}{2}S^{12} + \frac{x(x+1)(x+2)}{6}S^{111},$$

(79)

$$P_4(x; A) = xS^4 + \frac{x(x+5)}{2}S^{31} + \frac{x(x+3)}{2}S^{22} + \frac{x(x+1)}{2}S^{13}$$

$$+ \frac{x^3 + 6x^2 + 11x}{6}S^{211} + \frac{x^3 + 6x^2 + 5x}{6}S^{121}$$

$$+ \frac{x^3 + 3x^2 + 2x}{6}S^{112} + \frac{x(x+1)(x+2)(x+3)}{6}S^{1111}.$$

In particular, one has

(80)
$$\mathbf{ch}(\mathrm{PF}_n^{(r)}) = P_n(r; A).$$

But this characteristic can also be computed directly. Indeed, since $\mathrm{PF}_n^{(r)}$ is a permutational module, we have

(81)
$$\mathbf{ch}(\mathrm{PF}_n^{(r)}) = \sum_{I \vDash n} \alpha_I S^I,$$

where α_I is the number of nondecreasing words $\mathbf{a} \in \mathrm{PF}_n^{(r)}$ with packed evaluation I. These elements can be classified according to their parkized $\mathbf{b} = \mathrm{Park}(\mathbf{a})$ (see [18]), which is an ordinary nondecreasing parking function. The cardinality $\alpha_{\mathbf{b}}$ of such a class is a binomial coefficient. To see this, let

(82)
$$\mathbf{b} = \mathbf{b}_1 \bullet \mathbf{b}_2 \bullet \cdots \bullet \mathbf{b}_m$$

be the maximal factorization of $\mathbf{b}$ into connected nondecreasing parking functions ($\bullet$ denoting shifted concatenation, see [18]). The nondecreasing $\mathbf{a} \in \mathrm{PF}_n^{(r)}$ such that $\mathrm{Park}(\mathbf{a}) = \mathbf{b}$ are obtained by shifting each factor $\mathbf{b}_i$ of an amount k_i, such that $k_1 + \cdots + k_m \leq r$. Thus,

(83)
$$\alpha_{\mathbf{b}} = \binom{r+m-1}{m}.$$

Set $c(\mathbf{b}) = m$. Formula (83) being valid for all positive integers r, we have in general

(84)
$$P_n(x; A) = \sum_{I \vDash n} \left(\sum_{\mathbf{b} \in \mathrm{NDPF}_n;\ \mathrm{pEv}(\mathbf{b}) = I} \binom{x + c(\mathbf{b}) - 1}{c(\mathbf{b})} \right) S^I.$$

For example, there are three nondecreasing parking functions with packed evaluation (211): 1123, $1124 = 112 \bullet 1$ and $1134 = 11 \bullet 1 \bullet 1$, so that

(85)
$$\alpha_{211} = \binom{x}{1} + \binom{x+1}{2} + \binom{x+2}{3} = \frac{x^3 + 6x^2 + 11x}{6}.$$

Similarly, the coefficient of S^{31} in P_3 is

$$(86) \qquad \alpha_{31} = \binom{x}{1} + \binom{x}{1} + \binom{x+1}{2} = \frac{x(x+5)}{2},$$

corresponding to 1112, 1113 and $1114 = 111 \bullet 1$.

8.2.   By construction, the specialization $A = \mathbb{E}$ gives back the Abel polynomials. As usual, the specialization $A = 1$ is also interesting. Let $a(n, m)$ be the Catalan triangle [20, A009766]. That is,

$$(87) \qquad a(n, m) = \binom{n+m}{n} \frac{n-m+1}{n+1}$$

and

$$(88) \qquad \sum_{n \geq 0} \left(\sum_{m=0}^{n} a(n, m) t^m \right) z^n = \frac{C(tz)}{1 - zC(tz)},$$

where

$$(89) \qquad C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

is the generating series of the Catalan numbers. We need the reverted and shifted triangle

$$(90) \qquad c(n, k) = a(n-1, n-k-1) \ (n \geq 1), \quad c(0, 0) = 1,$$

whose generating series is

$$(91) \quad \frac{1}{1 - tzC(z)} = 1 + tz + (t^2 + t)z^2 + (t^3 + 2t^2 + 2t)z^3 + \cdots = \sum_{n \geq 0} \left(\sum_{m=0}^{n} c(n, m) t^m \right) z^n.$$

Let

$$(92) \qquad S_n(x) = \frac{x(x+1)\cdots(x+n-1)}{n!},$$

that is, $S_n(x)$ is the coefficient of t^n in $(1-t)^{-x}$ (which can be interpreted as $\sigma_t(x)$ for x a binomial element, whence the choice of notation).

We can now state:

Theorem 8.1. *The specialization $A = 1$ of the noncommutative Abel polynomials $P_n(x; A)$ is given by*

$$(93) \qquad P_n(x; 1) = \sum_{k=1}^{n} c(n, k) S_k(x).$$

Moreover, their generating series is

$$(94) \qquad \sum_{n \geq 0} P_n(x; 1) z^n = C(z)^x.$$

Proof – Equation (94) is clear if one rewrites the quadratic equation for $C(z)$ as

$$(95) \qquad C(z) = \frac{1}{1 - zC(z)} = \sum_{n \geq 0} S_n(1)[zC(z)]^n \,.$$

Equation (93) follows from (84), since (91) shows that $c(n, k)$ is the number of non-decreasing parking functions of length n such that $c(\mathbf{b}) = k$. It can also be proved analytically. The generating series of the right-hand sides of (93) can be written as a contour integral, over a circle $\gamma = \{|w| = \varepsilon < 1\}$

$$(96) \qquad \frac{1}{2\pi i} \oint_\gamma \frac{(1 - w)^{-x}}{w - zC(z)} dw = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - a} dw$$

where $a = zC(z)$ and $f(w) = (1 - w)^{-x}$. For $|z|$ small enough, a is inside γ, and by Cauchy's theorem, the right-hand side is

$$(97) \qquad f(a) = (1 - zC(z))^{-x} = C(z)^x$$

according to (95). ∎

The coefficients of the $P_n(x; 1)$ build up the triangle [20, A038455]. In fact, $C(z) = \mathcal{B}_2(z)$, where

$$(98) \qquad \mathcal{B}_t(z) = \sum_{n \geq 0} (tn)^{n-1} \cdot \frac{z^n}{n!}$$

is Lambert's generalized binomial series (see [5], (5.68) p. 200). According to [5], (5.70), we have finally the closed expression

$$(99) \qquad P_n(x; 1) = \binom{x + 2n}{n} \frac{x}{x + 2n} \,.$$

9. (k, l)-PARKING FUNCTIONS

There is a general notion of parking functions associated with a sequence $\mathbf{u} = (u_n)_{n \geq 1}$ of positive integers: these are the words $\mathbf{a}$ such that $(\mathbf{a}^\uparrow)_i \leq u_i$. In general, their enumeration can be obtained only in terms of Gončarov polynomials [14]. In the particular case where $\mathbf{u}$ is an arithmetic progression, it is possible to obtain closed formulas, of which we shall now give the noncommutative analogs.

Let

$$(100) \qquad \mathrm{PF}_n^{(k,l)} = \{\mathbf{a} \in [l + (n-1)k]^n \,|\, \mathbf{a}_i^\uparrow \leq l + (i-1)k\} \,.$$

Stanley and Pitman [21] have shown that

$$(101) \qquad |\mathrm{PF}_n^{(k,l)}| = l(l + kn)^{n-1} \,.$$

As above, this can be extended to the calculation of the 0-Hecke characteristic. The argument used for (16) proves as well (cf. [14])

$$(102) \qquad [N]^n = \bigsqcup_{j=0}^n \mathrm{PF}_j^{(k,l)} \,\shuffle\, [jk + l + 1, N]^{n-k} \,.$$

Taking $N = nk + r$, we obtain for the characteristic

$$(103) \qquad \sum_{n \geq 0} S_n((nk + r)A) = \left(\sum_{n \geq 0} \mathbf{ch} \, \mathrm{PF}_n^{(k,l)} \right) \left(\sum_{n \geq 0} S_n((nk + r - l)A) \right).$$

Setting

$$(104) \qquad F(r, k) = \sum_{n \geq 0} S_n((nk + r)A)$$

we have finally

Proposition 9.1. *The noncommutative characteristic of the permutational $H_n(0)$-module on $\mathrm{PF}_n^{(k,l)}$ is equal to $g_n^{(k,l)}(A)$, the term of degree n in $g^{(k,l)}(A) = F(r, k)F(r - l, k)^{-1}$, which is independent of r.*

Example 9.2. Taking $A = t\mathbb{E}$ and $r = l$, we recover the enumeration

$$(105) \qquad \sum_{n \geq 0} |\mathrm{PF}_n^{(k,l)}| \frac{t^n}{n!} = \frac{\sum_{n \geq 0} \frac{t^n}{n!}(nk + l)^n}{\sum_{n \geq 0} \frac{t^n}{n!}(nk)^n} = \sum_{n \geq 0} \frac{t^n}{n!} l(nk + l)^{n-1}$$

the last equality following from Abel's identity (the middle term is $g(tk\mathbb{E})^{l/k}$). Note that this can also be expressed in terms of the generalized exponential series of [5]

$$(106) \qquad \mathcal{E}_\alpha(z) = \sum_{n \geq 0} (n\alpha + 1)^{n-1} \frac{z^n}{n!},$$

that is,

$$(107) \qquad \sum_{n \geq 0} |\mathrm{PF}_n^{(k,l)}| \frac{t^n}{n!} = \mathcal{E}_{\frac{k}{l}}(lt).$$

If we set

$$(108) \qquad \mathcal{E}(z) = g(z\mathbb{E}),$$

we see that

$$(109) \qquad \mathcal{E}_{\frac{k}{l}} \left(\frac{l}{k}t \right) = \mathcal{E} \left(k\frac{t}{k} \right)^{\frac{l}{k}}.$$

Hence, for any α

$$(110) \qquad \mathcal{E}_\alpha(t) = \mathcal{E}(\alpha t)^{\frac{1}{\alpha}}$$

since this is true for α rational. This equality implies most of the interesting properties of $\mathcal{E}_\alpha(z)$. Indeed, let us write down explicitly the functional equation for $g(z\mathbb{E})$

$$(111) \qquad g(z\mathbb{E}) = \sum_{n \geq 0} S_n(z\mathbb{E}) g(z\mathbb{E})^n = \sum_{n \geq 0} \frac{z^n}{n!} g(z\mathbb{E})^n = e^{zg(z\mathbb{E})}.$$

We see that $\mathcal{E}(z)$ is Eisenstein's function, defined by

$$(112) \qquad \mathcal{E}(z) = e^{z\mathcal{E}(z)}$$

(see [5], (5.68) p. 200). Now, (110) implies immediately identities like [5], (5.69)

$$(113) \qquad \mathcal{E}_\alpha(z)^{-\alpha} \ln \mathcal{E}_\alpha(z) = z$$

for instance.

Example 9.3. The specialization $A = 1$ gives

$$(114) \qquad g_n^{(k,l)}(1) = \binom{nk+l+n-1}{n} \frac{l}{nk+l}.$$

This is defined for negative values of k and l as well, and we have

$$(115) \qquad \sum_{n\geq 0} g_n^{(-k,-l)}(1)(-t)^n = \sum_{n\geq 0} \binom{nk+l}{n} \frac{l}{nk+l} t^n = \mathcal{B}_k(t)^l$$

according to [5], (5.70). Again, this can be generalized. Recall that $\mathcal{B}_\alpha(z)$ is defined by

$$(116) \qquad \mathcal{B}_\alpha(z) = 1 + z \sum_{n\geq 0} \binom{(n+1)\alpha}{n} \frac{z^n}{n+1} = 1 + z \sum_{n\geq 0} \frac{\Lambda^n((n+1)\alpha)}{n+1} z^n,$$

(using the λ-ring notation), so that

$$(117) \qquad \mathcal{B}_{-\alpha}(-z) = 1 - z \sum_{n\geq 0} \frac{S^n((n+1)\alpha)}{n+1} z^n = 1 - zg(z\alpha),$$

where $g(z\alpha)$ denotes the specialization $A = z\alpha$ (α binomial and z of rank 1) of $g(A)$. Hence, $g(z\alpha)$ satisfies the functional equation

$$(118) \qquad g(z\alpha) = \sum_{n\geq 0} S_n(z\alpha)g(z\alpha)^n = (1 - zg(z\alpha))^{-\alpha},$$

so that

$$(119) \qquad g(z\alpha) = \mathcal{B}_{-\alpha}(-z)^{-\alpha}.$$

Clearly, this implies

$$(120) \qquad \mathcal{B}_{-\alpha}(-z)^{1+\alpha} - \mathcal{B}_{-\alpha}(-z)^\alpha = -z,$$

which is the first equation of [5, (5.69)]. Hence, the specialization $A = 1$ explains the properties of the generalized binomial series $\mathcal{B}_\alpha(z)$, while $A = \mathbb{E}$ takes cares of the generalized exponential series $\mathcal{E}_\alpha(z)$. Note that (119) allows one to write

$$(121) \qquad \mathcal{B}_\alpha(-z) = g(z\alpha)^{\frac{1}{\alpha}}$$

(but *not* $\mathcal{B}_\alpha(z) = g(-z\alpha)^{\frac{1}{\alpha}}$, since in the right-hand side, g is interpreted as a λ-ring operator!).

The q-characteristic is obtained similarly:

Proposition 9.4. *The q-characteristic of the permutational $H_n(0)$-module on $\mathrm{PF}_n^{(k,l)}$ admits as generating series*

$$\text{(122)} \qquad \sum_{n \geq 0} x^n q^{-k\binom{n+1}{2} - n(nk+l)} \mathbf{ch}_q \mathrm{PF}_n^{(k,l)} = F_k^{(r)}(x, q; A) F_k^{(r-l)}(q^l x, q; A)^{-1}$$

where

$$\text{(123)} \qquad F_k^{(r)}(x, q; A) = \sum_{n \geq 0} x^n q^{-k\binom{n+1}{2}} S_n([nk+r]_q A).$$

Example 9.5. Taking $A = 1$, $k = 3, l = 2$ and $r = \infty$, we obtain a q-analog of sequence [20, A069271] (1,2,9,52,340,2394,17710,...) for the q-enumeration of nondecreasing $\mathrm{PF}^{(3,2)}$'s (cf. [23], Cor. 5.1.)

(124)
$$f(t, q) = 1 + (q + 1)q^{-3}t + (q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1)q^{-9}t^2$$
$$+ (q^{12} + 3q^{11} + 5q^{10} + 7q^9 + 7q^8 + 7q^7 + 6q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1)q^{-18}t^3$$
$$+ O\left(t^4\right).$$

10. Generalized inversion formulas

10.1. **Some families of trees.** Let b be an integer. Consider the generalized inversion problem

$$\text{(125)} \qquad f = c + d_1 f^{b+1} + d_2 f^{b+2} + \dots$$

which reduces to Equations (32) and (55) for $b = 0$ and 1.

The general solution f_n is given by the Polish codes of ordered trees with $(b+1)n+1$ leaves and no vertices of arity between 1 and b.

We are interested in the special case

$$\text{(126)} \qquad g = 1 + d_1 g^{b+1} + d_2 g^{b+2} + \dots$$

that is, we want to compute the coefficients $\delta_I^{(b)}$ of

$$\text{(127)} \qquad g_n = \sum_{I \models n} \delta_I^{(b)} d^I.$$

Given the skeleton of a tree such that each internal vertex is of arity at least $b+1$, one can define its b-composition $I_b(S)$ as the sequence of values of the labels of the vertices of S minus b, in prefix order. Thanks to Equation (43), the number of trees with skeleton S is

$$\text{(128)} \qquad \prod_{k=1}^{p} \binom{i_k + b}{a_k}$$

where $I_b(S) = (i_1, \dots, i_p)$ and a_k is the arity of the k-th vertex of S in prefix order, so that, as in Equations (44) and (60):

$$(129) \qquad \delta_I^{(b)} = \sum_{(a_1,\dots,a_{p-1})} \prod_{k=1}^{p-1} \binom{i_k + b}{a_k}$$

where the sum is again taken over sequences $(a_1, \dots, a_{p-1})$ such that $a_1 + \cdots + a_j \geq j$ for all j and $a_1 + \cdots + a_{p-1} = p - 1$.

10.2. Combinatorial triangles. Let now $\gamma_{p,n}^{(b)}$ be

$$(130) \qquad \gamma_{p,n}^{(b)} := \sum_{I \models n; i_1 = p} \delta_I^{(b)}.$$

This amounts to enumerate the trees by arity of the root. The triangles $(\gamma_{p,n}^{(b)})$ include some classical triangles of the combinatorial literature: for $n = 0$, one recovers the Catalan triangle (sequence A033184 of [20]), for $n = 1$, one recovers the Schröder triangle (sequence A091370 of [20]). Their first terms are given on Figure 7.

1								1					
1	1							2	1				
2	2	1						7	3	1			
5	5	3	1					28	12	4	1		
14	14	9	4	1				121	52	18	5	1	
42	42	28	14	5	1			550	237	84	25	6	1
132	132	90	48	20	6	1		2591	1119	403	125	33	7	1

FIGURE 7. The Catalan and Schröder triangles ($b = 0$ and $b = 1$).

The triangles for $b = 2$ and $b = 3$ are given on Figure 8. Note that although they are not (yet) referenced in [20], the row sums of the case $b = 2$ yields Sequence A108447, with a quite different interpretation.

1								1					
3	1							4	1				
15	4	1						26	5	1			
85	22	5	1					192	35	6	1		
519	132	30	6	1				1531	270	45	7	1	
3330	837	190	39	7	1			12848	2215	362	56	8	1
22135	5516	1250	260	49	8	1		111818	18961	3054	469	68	9	1

FIGURE 8. Triangles obtained for $b = 2$ and $b = 3$.

One can also choose negative values for b even if this has no direct interpretation in terms of trees. With $b = -1$, the equation becomes

(131) $$f = c + d_1 + d_2 f + d_3 f^2 + \dots$$

and one recovers up to sign the Motzkin triangle (sequence A091836 of [20], see Figure 9) splitting up the Motzkin numbers (sequence A001006 of [20]) when putting $c = 1$ and considering d_i as a $(i-1)$-ary operation. Recall that Motzkin paths are the paths from $(0,0)$ to $(n,0)$, with three kinds of steps $(1,0)$, $(1,1)$, and $(1,-1)$, that never go below the horizontal axis.

$$
\begin{array}{ccccccc}
1 & & & & & & \\
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
2 & 3 & 3 & 1 & & & \\
4 & 6 & 6 & 4 & 1 & & \\
9 & 13 & 13 & 10 & 5 & 1 & \\
21 & 30 & 30 & 24 & 15 & 6 & 1
\end{array}
$$

FIGURE 9. The Motzkin triangle.

The bijection between trees and Motzkin paths is as follows: let P be a Motzkin path. Let $0 = i_1 < \dots < i_k = n$ be the sequence of abscissas of integer points $(i,0)$ belonging to P (also called the returns to zero of P). Denote by P_j the part of P between $(i_j, 0)$ and $(i_{j+1}, 0)$. Note that those elements have no non-trivial returns to zero.

Then the tree corresponding to P is built in the following recursive way: put k at the root of the tree (meaning d_k). If $P_j = (1,0)$ then put c as the j-th son of the root. Else, P_j is of the form $P_j = (1,1)Q_j(1,-1)$. Then insert Q_j recursively as the j-th son of the root.

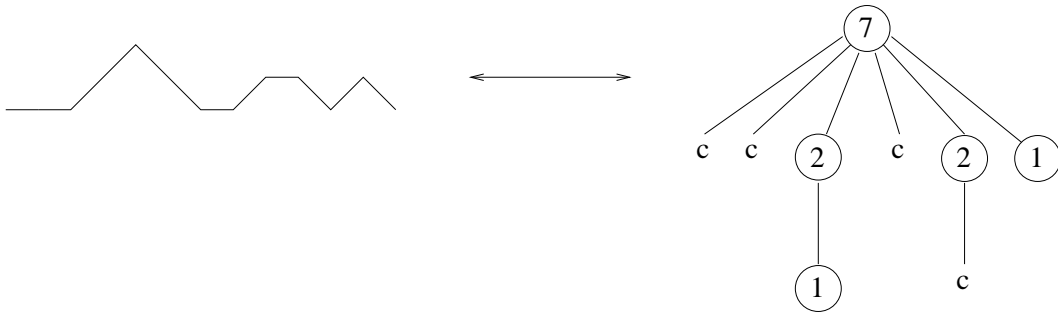Figure 10 presents an example of the bijection.

FIGURE 10. Example of the bijection between Motzkin paths and trees.

10.3. **Combinatorial sequences.** The generating functions of row-sums of the triangles are obtained by setting $d_i = t^i$ in Equation (126):

$$(132) \qquad\qquad g = 1 + \frac{tg^{b+1}}{1 - tg}.$$

For $b = 0$, one recognizes the quadratic equation satisfied by generating series of Catalan numbers and for $b = 1$ the quadratic equation of small Schröder numbers. For $b = -1$, $g(-t)$ satisfies the quadratic equation for the Motzkin numbers.

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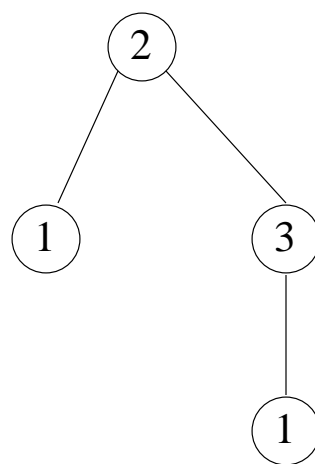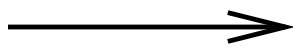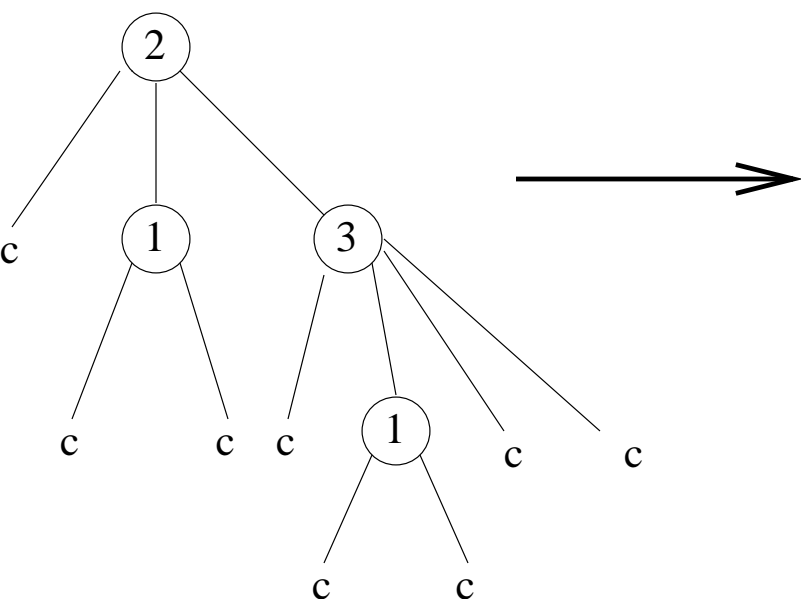[20] N.J.A. SLOANE, *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`

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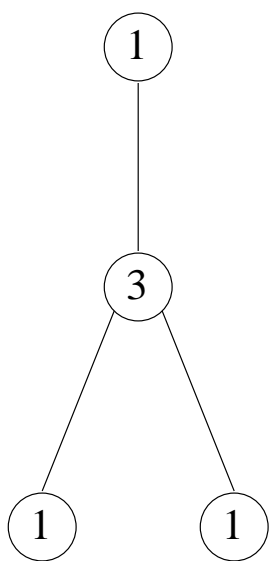*E-mail address*, Jean-Christophe Novelli: `novelli@univ-mlv.fr`

E-mail address, Jean-Yves Thibon: `jyt@univ-mlv.fr`

$$f \quad = \quad c \quad + \quad \underset{\substack{c \qquad c}}{\overset{\textcircled{1}}{\diagup\diagdown}} \quad + \quad \underset{\substack{c \qquad \underset{\substack{c \qquad c}}{\textcircled{1}}}}{\overset{\textcircled{1}}{\diagup\diagdown}} \quad + \quad \underset{\substack{\underset{\substack{c \qquad c}}{\textcircled{1}} \qquad c}}{\overset{\textcircled{1}}{\diagup\diagdown}} \quad + \quad \underset{\substack{c \quad c}}{\overset{\textcircled{2}}{\diagup|\diagdown}}$$

16   12   4   2   0

```
        113355
     2           4
112255        113344
     4           2
        112244
           3
        112233
```

```
        122446
   5  /        \  3
     /          \
122445        122336
      \  3    5  /
       \      /
        122335
          |  4
          |
        122334
```